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Bireflectivity

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Abstract

Motivated by a model for syntactic control of interference, we introduce a general categorical concept of bireflectivity. Bireflective subcategories of a category $\mathcal{A}$ are subcategories with left and right adjoint equal, subject to a coherence condition. We characterize them in terms of split-idempotent natural transformations on $\mathrm{id}_{\mathcal{A}}$. In the special case that $\mathcal{A}$ is a presheaf category, we characterize them in terms of the domain, and prove that any bireflective subcategory of $\mathcal{A}$ is itself a presheaf category. Given a small symmetric monoidal category $\mathcal{C}$, we define diagonal structure on $\mathcal{C}$, which is that structure and a little less than those axioms required to prove the monoidal structure is finite product structure. We then obtain a bireflective subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and deduce results relating its finite product structure with the monoidal structure of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ determined by that of $\mathcal{C}$. We also investigate closed structure.

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1 Introduction

This paper is a companion paper to *Syntactic Control of Interference Revisited* [10] in this volume. In this paper, we introduce a general categorical concept, *bireflectivity*, to analyse the properties of the model of the SCIR type system given in [10]. This paper is purely categorical: it can be read independently of [10] as a category theoretic paper. The bireflectivity concept has much wider applicability, but this paper concentrates on our leading example taken from [10]; although we will describe it again here, we will not explain its significance.

The central surprising category theoretic feature of the model of SCIR given in [10] is the concept of a "bireflective" subcategory, by which we mean a subcategory with inclusion having both left and right adjoint, with those adjoints equal, and satisfying an evident coherence condition relating the unit and counit. In [10], the one and only nontrivial bireflective subcategory of the semantic category is the subcategory of *passive* objects. For many categories, such as **Set**, **Poset**, and the category of ω-cpo's, there is no nontrivial such subcategory, and in fact, we prove that any well pointed category has no nontrivial such subcategory.

In this paper, we characterize bireflective subcategories of a category $\mathcal{A}$ as equivalent to split-idempotent natural transformations from the identity functor on $\mathcal{A}$ to itself. The construction implicit in this result uses a limit in the 2-category **Cat**, called an *identifier*. So we describe the notion of identifier, give the construction, and prove our result. In the particular case that $\mathcal{A}$ is a presheaf category $[\mathcal{A}'^{\mathrm{op}}, \mathbf{Set}]$, we prove more: that any bireflective subcategory $\mathcal{B}$ of $\mathcal{A}$ must itself be a presheaf category; and we give an explicit description of a $\mathcal{B}'$ for which $\mathcal{B} = [\mathcal{B}'^{\mathrm{op}}, \mathbf{Set}]$.

The semantic category $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ of [10] is a mild variant of a presheaf category, and for our purposes, satisfies the same conditions. So although we study presheaf categories in this paper, it is routine to verify that our analysis all extends to $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$. Specifically, if $\mathcal{A}'$ is a small monoidal category, then $[\mathcal{A}'^{\mathrm{op}}, \mathbf{Set}]$ is the free monoidal cocompletion of $\mathcal{A}'$ [3,6,10]. With a little more structure on $\mathcal{A}'$, which we call *diagonal* structure, we can construct an idempotent natural transformation from $\mathrm{id}_{\mathcal{A}'}$ to itself, and hence a split one from $\mathrm{id}_{[\mathcal{A}'^{\mathrm{op}}, \mathbf{Set}]}$ to itself, thus yielding a bireflective subcategory of $[\mathcal{A}'^{\mathrm{op}}, \mathbf{Set}]$. For an example, in [10], the category of worlds is a small monoidal category with diagonal structure, and generalizing mildly from **Set** to the category of domains, our construction yields the monoidal structure on the semantic category $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ of [10] and its restriction to the passive objects. Here, we use diagonal structure to deduce several results about the interaction of the bireflective subcategory $\mathcal{B}$ with $\mathcal{A}$: both adjunctions between them are monoidal adjunctions; $\mathcal{B}$ has finite products given by the restriction of the tensor product on $\mathcal{A}$; $\mathcal{B}$ is contained in the category of commutative comonoids on $\mathcal{A}$; and $\mathcal{B}$ is an exponential ideal of $\mathcal{A}$. These results are central to the analysis of [10].

For this paper, we do not make heavy use of 2-categories beyond their definition; a standard reference to an analysis of the definition is [9] by Kelly and Street.

2 The semantic category

In this section, we recall the semantic definitions of [10].

Definition 2.1 (**The category of worlds**) The category $\mathbf{X}$ has as objects countable sets, with a morphism (f, R) from X to Y given by a function $f : X \longrightarrow Y$ and an equivalence relation R on X such that $xRy \ \wedge \ fx = fy \Rightarrow x = y$; and with composition $(g, S) \circ (f, R)$ given by the function $g \circ f$ and the relation T, where $xTy \Leftrightarrow xRy \wedge fxSfy$

Proposition 2.2 *Finite product of sets gives a symmetric monoidal structure on $\mathbf{X}$ with unit the terminal object.*

Proof. For $(f, R) : X \longrightarrow Z$ and $(g, S) : Y \longrightarrow W$ in $\mathbf{X}$, the tensor $(f, R) \otimes (g, S)$ is given by $(f \times g, R \times S) : X \times Y \longrightarrow Z \times W$, where $(x, y)(R \times S)(x', y') \Leftrightarrow xRx' \wedge ySy'$. The canonical isomorphisms are given by those of finite products with total relations. The singleton set 1 is terminal in $\mathbf{X}$ with the unique morphism t_W from W given by the unique function and the equality relation on W. $\square$

Definition 2.3 Given $W \in \mathbf{X}$, define the *state change constraint* morphism $\alpha_W : W \longrightarrow W$ by the identity function and the identity on W.

Note that α is an idempotent natural transformation on $\mathrm{id}_{\mathbf{X}}$. We write $\pi_0 : X \otimes Y \longrightarrow X$ for $X \otimes t_Y$ and similarly for $\pi_1 : X \otimes Y \longrightarrow X$.

In [10], the type theory SCIR is modelled in the *semantic category* $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$, where $\mathbf{D}$ is the category of possibly bottomless ω-complete posets and continuous functions.

Definition 2.4 *(***Passive objects***)* For $f \in [\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$, f is called *passive* if $f\alpha = \mathrm{id}_f : f \Longrightarrow f : \mathbf{X}^{\mathrm{op}} \longrightarrow \mathbf{D}$. The full subcategory $\mathbf{P}$ of $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ is given by the passive objects. The full inclusion is written $J : \mathbf{P} \longrightarrow [\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$.

Definition 2.5 Define a monoidal structure on $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ as follows. For $f, g \in [\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$, the tensor product $f \otimes g$ is given by $(f \otimes g)W = \{(a, b) \mid a \in fW, b \in gW, a\#b\}$, where

$$\begin{aligned} a\#b \Leftrightarrow\ & \exists u : W \longrightarrow X \otimes Y \text{ in } \mathbf{X}\ . \\ & \exists a' \in fX.\ \exists b' \in gY. \\ & a = f(\pi_0 u)a' \wedge b = g(\pi_1 u)b', \end{aligned}$$

and, for $u : W \longrightarrow Z$ in $\mathbf{X}$, $(f \otimes g)(u)(a, b) = (f(u)a, g(u)b)$. The unit is the terminal object of $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$.

Through the course of this paper, we prove the following properties of the category of passive objects used in [10].

Proposition 2.6 *The full subcategory* $\mathbf{P}$ *is both reflective and coreflective in the semantic category* $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$*; moreover, the reflector and coreflector coincide.* $\square$

Proposition 2.7 *The category* $\mathbf{P}$ *has finite products.* $\square$

Proposition 2.8 *The symmetric monoidal structure on* $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ *restricts to the cartesian structure on* $\mathbf{P}$. $\square$

Proposition 2.9 *Both the inclusion and the reflector (=coreflector) are strong symmetric monoidal functors, i.e., they preserve the monoidal structure.* $\square$

Proposition 2.10 $\mathbf{P}$ *is an exponential ideal of* $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$*, i.e., given* $P \in \mathbf{P}$ *and* $A \in [\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$*, the exponential object* $[A, P]$ *lies in* $\mathbf{P}$. $\square$

Of course, one could prove these results directly, rather than by appeal to the abstract theory we develop here. However, it seems likely that other models of syntactic control of interference will be developed in future, so rather than having to prove such results every time one discovers a new model, it seems useful to have a general result from which one can deduce them automatically. Moreover, our general results provide necessary and sufficient conditions for the natural level of generality of the arguments, so they set parameters to the search for models that satisfy the properties we study.

Remark 2.11 Robin Cockett and Robert Seely have pointed out (personal communication) that a second tensor can be defined on the category $\mathbf{X}$ of worlds: on objects it yields the disjoint union of the sets, and on morphisms, yields the sum of the function parts and the "join" of the equivalence-relation parts. The second tensor also lifts to the semantic category $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ which, together with the bireflective subcategory $\mathbf{P}$, provides an example of a weakly distributive model of negation-free linear logic [2], with ! and ? both given by the bireflector. This construction cannot be non-trivially generalized to model *full* linear logic, for if the semantic category were $*$-autonomous, the bireflective subcategory (which is both the category of algebras for ? and the category of co-algebras for !) would be both cartesian closed and co-cartesian closed, and hence degenerate.

3 Bireflectivity

In this section, we define the notion of bireflective subcategory and characterize bireflective subcategories in a given category. After giving a few examples, we use this characterization to show that any bireflective subcategory of a presheaf category is itself a presheaf category. So in particular, the category of passive objects of the previous section is a presheaf category. In fact, it follows from our analysis that it is the *only* nontrivial bireflective subcategory of $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$.

Definition 3.1 A *bireflective* subcategory of a category $\mathcal{A}$ is a subcategory $\mathcal{B}$ of $\mathcal{A}$ with inclusion $J : \mathcal{B} \longrightarrow \mathcal{A}$ that has left and right adjoints equal, say

$S : \mathcal{A} \longrightarrow \mathcal{B}$, with

$$\begin{tikzcd} JSA \arrow[r, "\varepsilon'_A"] \arrow[dr, "\mathrm{id}"'] & A \arrow[d, "\eta_A"] \\ & JSA \end{tikzcd}$$

commuting, where η is the unit of adjunction $S \dashv J$ and ε' is the counit of $J \dashv S$.

Proposition 3.2 *Any bireflective subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is full, closed under subobject formation, and closed under quotient formation.*

Proof. With notation as in Definition 3.1, for $B \in \mathcal{B}$, $\eta_{JB} = \eta_{JB}\varepsilon'_{JB}J\eta'_B = J\eta'_B$. So, for any $f{:}JB \longrightarrow JC$ in $\mathcal{A}$, $f = J(\overline{f}\eta'_B)$ with $\overline{f}$ the transposition of f under $S \dashv J$. For subobject formation, let $m : A \longrightarrow JB$ be a monomorphism with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We show that $\varepsilon'_A\eta_A = \mathrm{id}_A$. This is equivalent to $m\varepsilon'_A\eta_A = m$; using the coherence condition, $m = J(\overline{m})\eta_A = J(\overline{m})\eta_A\varepsilon'_A\eta_A = m\varepsilon'_A\eta_A$, with $\overline{m}$ the transposition of m. Closure under quotient formation is proved dually. $\square$

In this and subsequent sections, endo natural transformations whose components are all split idempotents play a central role. We call such a natural transformation a *split-idempotent natural transformation.*

Theorem 3.3 *Given a category $\mathcal{A}$, to give a bireflective subcategory of $\mathcal{A}$ is to give a split-idempotent natural transformation on* $\mathrm{id}_{\mathcal{A}}$.

In order to prove this, we need the construction of a bireflective subcategory from a split-idempotent natural transformation on $\mathrm{id}_{\mathcal{A}}$. This is given by a limit in the 2-category **Cat**, called an *identifier.*

Definition 3.4 Let $\mathcal{K}$ be a 2-category and $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}$ be a 2-cell in it. The *identifier* of α is the universal 1-cell $h : Z \longrightarrow X$ such that $fh = gh$ and $\alpha h = \mathrm{id} : fh \Longrightarrow gh$.

Spelling this out, h has two properties:

(i) given $k{:}W \longrightarrow X$ such that αk is an identity 2-cell, there exists a unique 1-cell $k^\alpha : W \longrightarrow Z$ such that

$$\begin{tikzcd} Z \arrow[r, "h"] & X \\ W \arrow[u, dashed, "k^\alpha"] \arrow[ur, "k"'] & \end{tikzcd}$$

commutes.

(ii) given $k, k' : W \longrightarrow X$ with αk and $\alpha k'$ both identities, and $\beta : k \Longrightarrow k'$, there exists a unique 2-cell $\beta^\alpha : k^\alpha \Longrightarrow k'^\alpha$ such that $h\beta^\alpha = \beta$.

Identifiers are limits in 2-categories, as explained in Kelly's article [8].

Proof (of Theorem 3.3). Let $\alpha : \mathrm{id}_\mathcal{A} \Rightarrow \mathrm{id}_\mathcal{A} : \mathcal{A} \longrightarrow \mathcal{A}$ be a split-idempotent natural transformation, $J : \mathcal{A}^\alpha \longrightarrow \mathcal{A}$ the identifier of α, and $\mathrm{id}_\mathcal{A} \overset{r}{\Rightarrow} R \overset{j}{\Rightarrow} \mathrm{id}_\mathcal{A}$ the splitting of α. Then $\alpha R = \mathrm{id}_R$. By the universality of the identifier J, one has a unique functor $S : \mathcal{A} \longrightarrow \mathcal{A}^\alpha$ with $R = JS$. The adjunction $S \dashv J$ is given by

$$\begin{array}{rcl} \mathcal{A}^\alpha(SA, B) & \cong & \mathcal{A}(A, JB) \\ f & \mapsto & Jf \circ r_A \\ J^{-1}_{SA,B}(g \circ j_A) & \leftarrow\!\shortmid & g \end{array}$$

with unit r. Applying the same argument to $\mathcal{A}^{\mathrm{op}}$, one obtains $J \dashv S$ with counit j. For the reverse direction, $\varepsilon' \cdot \eta$ gives the desired split idempotent. It is easy to verify these constructions are mutually inverse. $\square$

Theorem 3.3 allows one to replace an analysis of bireflective subcategories by that of split-idempotent natural transformations, which is often easier.

Example 3.5 The category of finite semilattices is bireflective in the category of finite commutative semigroups. First note that any one-generator finite commutative semigroup G has exactly one idempotent. With the additive notation, let G be generated by x with relation $ix = (i+k)x$ $(i, k > 0)$. There is a unique h with $0 \le h < k$ and $k|(i+h)$, say $nk = i+h$. Since $(i+h)x + kx = (i+k)x + hx = (i+h)x$, one has $2(i+h)x = (i+h)x + nkx = (i+h)x$, i.e., $(i+h)x$ is an idempotent. For unicity, if jx is also an idempotent, $jx = (i+h)(jx) = j((i+h)x) = (i+h)x$. Given a finite commutative semigroup G and $x \in G$, let x' be the unique idempotent in $<x> \subset G$, the finite subsemigroup of G generated by x. The function $\alpha_G : x \mapsto x'$ is an endomorphism on G since given $x, y \in G$, $x' + y'$ is an idempotent in $<x+y> \subset G$, and hence $(x+y)' = x' + y'$ by the uniqueness. The uniqueness also implies that α_G is natural in G. Finally, α_G splits with the retract $\{x \in G \mid x + x = x\}$, which is a semilattice with the order $x \le y \iff x + y = y$. Similarly, semilattices in the category of torsion commutative semigroups form a bireflective subcategory. One may also replace semigroups by monoids.

Example 3.6 The category **Rel** of sets and relations is bireflective in **SProc**, the interaction category of synchronous processes ([1], Proposition 3.0.4). Briefly, an object A of **SProc** is a pair (Σ_A, S_A) of sets with S_A a nonempty prefix closed subset of Σ_A^*; a morphism from A to B is a strong bisimilar class of $\Sigma_A \times \Sigma_B$-labeled transition systems whose traces are contained in $S_A \times S_B$ in the obvious sense; the composite $(P;Q) : A \xrightarrow{P} B \xrightarrow{Q} C$ is given by "synchronization" at B, i.e., for $(a, c) \in \Sigma_A \times \Sigma_C$, there is a $(P;Q)$-transition $(p;q) \xrightarrow{(a,c)} (p';q')$ if and only if there exists $b \in \Sigma_B$ with a P-transition $p \xrightarrow{(a,b)} p'$ and a Q-transition $q \xrightarrow{(b,c)} q'$; and finally, the identity on A is given by the $\Sigma_A \times \Sigma_A$-labeled transition system whose traces are $\{$"$(a_0, a_0)(a_1, a_1)\cdots$" $\mid$ "$a_0 a_1 \cdots$" $\in S_A\}$. Given an object A, let $S_A|_n$ be the subset of S_A given by the strings of length at most n. There is a trivial, one-step $\Sigma_A \times \Sigma_A$-labeled transition system α_A with 'start' $\xrightarrow{(a,b)}$ 'end' $\Leftrightarrow a = b \wedge a \in S_A|_1$. For $P : A \longrightarrow B$, both $\alpha_A;P$ and $P;\alpha_B$ are bisimilar to the transition system P "truncated" to at most one-step. So $\alpha_A : A \longrightarrow A$ is natural in $A \in$ **SProc**. This also splits, giving the retract $((S_A|_1)', S_A|_1)$, where $(S_A|_1)'$ is $S_A|_1$ minus the empty string. The statement at the beginning holds since the full subcategory of **SProc** given by those A with α_A the identity transition system is precisely **Rel**.

Example 3.7 Let $\mathcal{A}$ be a topos and $\mathrm{Rel}(\mathcal{A})$ the category of relations in $\mathcal{A}$; i.e., $\mathrm{Rel}(\mathcal{A})$ has the same objects as $\mathcal{A}$ and a $\mathrm{Rel}(\mathcal{A})$-morphism is an equivalence class of monic pairs, with the composition given by pullbacks and the epi-mono factorization. Let $\mathcal{B}$ be a collection of $\mathcal{A}$-objects that includes the terminal object. Then, $\mathrm{Rel}(\mathcal{B})$, the full subcategory of $\mathrm{Rel}(\mathcal{A})$ given by the objects in $\mathcal{B}$, is bireflective in $\mathrm{Rel}(\mathcal{A})$ if and only if $\mathcal{B}$, as a full subcategory, is reflective in $\mathcal{A}$ and closed under power object formation. This requires a non-trivial proof, but the proof rests on the fact that certain split-idempotent natural transformations on $\mathrm{id}_{\mathrm{Rel}(\mathcal{A})}$ are necessarily equivalence relations. Full subcategories of a topos closed under power object formation are studied in detail by Freyd [5] with an application in logic.

We plan to prepare a sequel giving full details of these applications and other characterizations of bireflectivity. We now return to our leading example.

Given a 2-category $\mathcal{K}$, a coidentifier in $\mathcal{K}$ is an identifier in $\mathcal{K}^{\mathrm{op}}$, reversing the 1-cells in the definition. For our main example of a coidentifier,

Example 3.8 Let $\mathcal{C}$ be a category, and let $\alpha : \mathrm{id} \Longrightarrow \mathrm{id} : \mathcal{C} \longrightarrow \mathcal{C}$ be an idempotent natural transformation. Then, the coidentifier $\mathcal{C}_\alpha$ is given by factoring $\mathcal{C}$ by the congruence $\sim$, where for $f, g : A \longrightarrow B$,

$$f \sim g \Leftrightarrow \alpha_B \cdot f = \alpha_B \cdot g.$$

To see this, first observe that $\sim$ is a congruence on $\mathcal{C}$: it is obviously an equivalence on each hom-set $\mathcal{C}(A, B)$; it respects composition in $\mathcal{C}$ because α is natural. Now if α is identified with the identity and $f \sim g$, then f is identified with $\alpha_B \cdot f = \alpha_B \cdot g$, which is identified with g. Conversely, $\alpha_B \cdot \mathrm{id}_B = \alpha_B = \alpha_B \cdot \alpha_B$, so $\mathrm{id}_B \sim \alpha_B$.

So, we may describe the coidentifier $\mathcal{C}_\alpha$ of α by

$$\begin{aligned} &\mathrm{Ob}\,(\mathcal{C}_\alpha) = \mathrm{Ob}\,(\mathcal{C}) \\ &\mathcal{C}_\alpha(A, B) = \mathcal{C}(A, B)/\!\sim, \qquad \textit{where } f \sim g \textit{ if and only if } \alpha_B \cdot f = \alpha_B \cdot g. \end{aligned}$$

Given any functor $h : \mathcal{A} \longrightarrow \mathcal{B}$ between small categories, one has a functor $[h, \mathcal{D}] : [\mathcal{B}, \mathcal{D}] \longrightarrow [\mathcal{A}, \mathcal{D}]$. If $\mathcal{D}$ is complete and cocomplete, $[h, \mathcal{D}]$ has left and right adjoints given by left and right Kan extension. If $h : \mathcal{A} \longrightarrow \mathcal{B}$ is the coidentifier of a natural transformation between functors whose value is equal on objects, it follows from the universal property that $[h, \mathcal{D}]$ is fully faithful, exhibiting $[\mathcal{B}, \mathcal{D}]$ as equivalent to a full subcategory of $[\mathcal{A}, \mathcal{D}]$. That subcategory is given by those $f : \mathcal{A} \longrightarrow \mathcal{D}$ such that $f\alpha = \mathrm{id}$, where α is the natural transformation.

Proposition 3.9 *Given a category $\mathcal{D}$ where every idempotent splits and a natural idempotent $\alpha : \mathrm{id} \Longrightarrow \mathrm{id} : \mathcal{C} \longrightarrow \mathcal{C}$, the full inclusion $[h, \mathcal{D}] : [\mathcal{C}_\alpha, \mathcal{D}] \longrightarrow [\mathcal{C}, \mathcal{D}]$ exhibits $[\mathcal{C}_\alpha, \mathcal{D}]$ as a bireflective subcategory of $[\mathcal{C}, \mathcal{D}]$. The adjoint of $[h, \mathcal{D}]$ takes f to the splitting of $f\alpha : f \Longrightarrow f$.*

Proof. This follows by using $[-, \mathcal{D}] : \mathbf{Cat}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ to send colimits in **Cat** to limits, hence coidentifiers to identifiers, and by applying the construction of Theorem 3.3. Note that $[\alpha, \mathcal{D}]$ always splits. □

Definition 3.10 A fully faithful functor $Z : \mathcal{G} \longrightarrow \mathcal{C}$ is *generating* if the functor $\tilde{Z} : \mathcal{C} \longrightarrow [\mathcal{G}^{\mathrm{op}}, \mathbf{Set}]$, $C \mapsto \mathcal{C}(Z-, C)$, is faithful.

Spelling this out, a full subcategory $\mathcal{G}$ of $\mathcal{C}$ generates $\mathcal{C}$ if for any parallel pair of distinct maps $f, g : A \longrightarrow B$ in $\mathcal{C}$, there exists an object X of $\mathcal{C}$ and a map $h : X \longrightarrow A$ such that fh and gh are distinct. For example, the unit category $\{1\}$ is generating in **Set** and in **Poset**, and the arrow category is generating in **Cat**.

A category $\mathcal{C}$ with a terminal object 1 is *well-pointed* if the inclusion $\{1\} \longrightarrow \mathcal{C}$ is generating.

Proposition 3.11 *Given a generating functor $Z : \mathcal{G} \longrightarrow \mathcal{C}$, any endo natural transformation on $\mathrm{id}_\mathcal{C}$ is uniquely determined by its restriction to Z.*

Proof. Given $s, t : \mathrm{id}_\mathcal{C} \Longrightarrow \mathrm{id}_\mathcal{C}$, $sZ = tZ$ implies for each $C \in \mathcal{C}$, $G \in \mathcal{G}$, and $f : ZG \longrightarrow C$, by naturality, $s_C f = f s_{ZG} = f t_{ZG} = t_C f$. □

So, there are no more endo natural transformations on $\mathrm{id}_\mathcal{C}$ than there are on Z.

Corollary 3.12 *For a well-pointed category $\mathcal{C}$, there is no nontrivial idempotent natural transformation on $\mathrm{id}_\mathcal{C}$.*

Proof. The only natural transformation on the inclusion of $\{1\}$ is the identity on 1. □

Remark 3.13 In our category $\mathbf{X}$ of worlds (Definition 2.1), $\{1\}$ is not a generator (as one cannot distinguish two morphisms which differ only in their equivalence relation parts), but $\{2\}$, the one object subcategory of $\mathbf{X}$ given by the two element set 2, is. Applying the above proposition, there are at most six natural transformations on $\mathrm{id}_\mathbf{X}$, of which four can be idempotent. By examining each, one can conclude $\alpha_X = (\mathrm{id}_X, \Delta_X)$ is the only idempotent natural transformation on $\mathrm{id}_\mathbf{X}$ other than the identity.

We may use Remark 3.13 to deduce that our semantic category $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ has only one nontrivial bireflective subcategory, which is of course the subcategory of passive objects. In fact, we show a stronger result: to give an (split-)idempotent natural transformation on $\mathrm{id}_{[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]}$ is to give an idempotent natural transformation on $\mathrm{id}_\mathcal{C}$. This gives a converse to Proposition 3.9

in case the base category is **Set**. The lifting of this result from **Set** to **D** is routine: we give it for **Set** for ease of exposition.

Proposition 3.14 *For a small category $\mathcal{C}$, to give an idempotent natural transformation on* $\mathrm{id}_{[\mathcal{C}^{\mathrm{op}},\mathbf{Set}]}$ *is to give one on* $\mathrm{id}_{\mathcal{C}}$.

Proof. Given an idempotent natural transformation $\alpha : \mathrm{id}_{\mathcal{C}} \Longrightarrow \mathrm{id}_{\mathcal{C}}$, it extends to $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ by homming α^{op} into **Set**. Now given any idempotent natural transformation $\beta : \mathrm{id}_{[\mathcal{C}^{\mathrm{op}},\mathbf{Set}]} \Longrightarrow \mathrm{id}_{[\mathcal{C}^{\mathrm{op}},\mathbf{Set}]}$, by the fact that every $F : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a colimit of representables, β is fully determined by its behaviour on representables. Thus every such β arises from a unique $\alpha : \mathrm{id}_{\mathcal{C}} \Longrightarrow \mathrm{id}_{\mathcal{C}}$. □

Example 3.15 Given a monoid M with zero element 0 ($x0 = 0x = 0$ for all $x \in M$), **Set** is bireflective in the category of M-sets, corresponding to the idempotent 0 on id_M with M regarded as a one object category.

Remark 3.16 Bireflectivity seems to be the distinctive categorical property that differentiates the model based on the category $\mathbf{X}$ from other extant examples of functor category semantics. For example, Oles originally used a full on objects subcategory $\mathbf{X}'$ of $\mathbf{X}$, the maps (f, R) being those where the restriction of f to any R-equivalence class is bijective. Clearly, this rules out the state change constraint endomorphisms α_W, and so there is only one natural idempotent on $\mathrm{id}_{\mathbf{X}'}$, namely the identity. As a result, the functor category used by Oles possesses no nontrivial bireflective subcategories.

4 Diagonal Categories

In this section, we define *diagonal* structure on a symmetric monoidal category. A diagonal structure consists of the data and some of the axioms required to force the monoidal structure to be finite product structure. Of course, the category of worlds $\mathbf{X}$ has diagonal structure, as does any category with finite products. From diagonal structure, one can obtain an idempotent natural transformation that, in a precise sense, measures the extent to which the diagonal structure fails to be finite product structure. This idempotent allows us to define a bireflective subcategory of the presheaf category as in the previous section, and the diagonal structure further allows us to deduce results such as that the monoidal structure on the presheaf category restricts to finite product structure on the bireflective subcategory, and that the adjunction becomes a monoidal adjunction.

Definition 4.1 A *diagonal* category is a symmetric monoidal category $\mathcal{C}$ whose unit is the terminal object of $\mathcal{C}$, together with a natural transformation with components $\delta_A : A \longrightarrow A \otimes A$, called the *diagonal* morphism on A, such that

$$\begin{tikzcd}
A \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & A \otimes A \arrow[d, "A \otimes \delta_A"] \\
A \otimes A \arrow[r, "\delta_A \otimes A"'] & A \otimes A \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[dl, "\delta_A"'] \arrow[dr, "\delta_A"] & \\
A \otimes A \arrow[rr, "c"'] & & A \otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
& A \otimes B \arrow[dl, "\delta_{A\otimes B}"'] \arrow[dr, "\delta_A \otimes \delta_B"] & \\
A \otimes B \otimes A \otimes B \arrow[rr, "A \otimes c \otimes B"'] & & A \otimes A \otimes B \otimes B
\end{tikzcd}$$

commute.

It is routine to verify that in a diagonal category $\mathcal{C}$, the maps $(t \otimes A) \cdot \delta_A$ form an idempotent natural transformation from $\mathrm{id}_{\mathcal{C}}$ to $\mathrm{id}_{\mathcal{C}}$.

Our leading example of diagonal structure is as follows.

Example 4.2 On $\mathbf{X}$, define $\delta_A : A \longrightarrow A \otimes A$ by the diagonal together with the total relation.

Example 4.3 Consider the symmetric monoidal closed category $\mathbf{Set}_*$ of pointed sets, the monoidal structure being smash product. A $\mathbf{Set}_*$-category is a category with zero morphisms. Consider any $\mathbf{Set}_*$-category with finite products. Then, the finite products define the symmetric monoidal structure and we may define $\delta_A : A \longrightarrow A \times A$ to be the zero morphism. Specific examples of such categories are the categories of monoids, of pointed sets, and of ω-cpo's with bottom and bottom preserving maps.

Example 4.4 Any category with finite products.

It is easy to see that in Examples 4.2 and 4.3, the structure is not that of finite products since the diagram

$$\begin{tikzcd}
A \arrow[r, "\delta_A"] \arrow[dr, "\mathrm{id}"'] & A \otimes A \arrow[d, "t \otimes A"] \\
& A
\end{tikzcd}$$

does not commute. Observe that in Example 4.2, $(t \otimes A) \cdot \delta_A$ is the state change constraint idempotent α_A. Also in Example 4.2 and the three specific examples in 4.3, $(t \otimes A) \cdot \delta_A$ is the only nontrivial idempotent natural transformation on the identity functor by Proposition 3.11.

Proposition 4.5 *The data for a diagonal category form finite product struc-*

ture if and only if

$$\begin{tikzcd} A \arrow[r, "\delta_A"] \arrow[dr, "\mathrm{id}"'] & A \otimes A \arrow[d, "t \otimes A"] \\ & A \end{tikzcd}$$

commutes.

Proof. ($\Leftarrow$) In any category with finite products, the composite of the diagonal with the projection must be the identity.
($\Rightarrow$) Given $(f : C \longrightarrow A, g : C \longrightarrow B)$, define $h : C \longrightarrow A \times B$ to be

$$C \xrightarrow{\delta_C} C \otimes C \xrightarrow{f \otimes g} A \otimes B.$$

It is routine to verify, using the equation and the terminal object condition, that the appropriate two diagrams commute. Unicity is similar, using the third of the three diagonal commutativities. □

Proposition 4.6 *Let $\mathcal{C}$ be a diagonal category. Then the free category on $\mathcal{C}$ that forces the diagonal data of $\mathcal{C}$ to be finite products, is given by the coidentifier of the natural transformation determined by $(t \otimes A) \cdot \delta_A : A \longrightarrow A$.*

Proof. This follows from Proposition 4.5 because sending the diagonal data to finite product structure necessitates the identification of $(t \otimes A) \cdot \delta_A : A \longrightarrow A$ with id_A, and such identification with the addition of no further objects or arrows yields a finite product structure. □

Example 4.7 Applying the construction of Proposition 4.6 to Example 4.2 yields the category of countable sets. For Example 4.3, the construction of 4.6 yields a category equivalent to the unit category. The construction of the category $\mathbf{X}$ of worlds from the category of countable sets generalizes easily to a construction on any small category with finite limits. One still acquires a diagonal category, and following that construction by that of Proposition 4.6 returns the original category.

To end this section, we digress briefly to observe that, for general reasons, *any* monoidal structure on $\mathbf{X}$ giving rise to an idempotent natural transformation on $\mathrm{id}_{\mathbf{X}}$ is restricted. The argument goes as follows.

Proposition 4.8 (Foltz, Kelly, and Lair [4]) *Given a monoidal category $\mathcal{C}$ with unit I, any idempotent f on I extends to an idempotent natural transformation f^* on $\mathrm{id}_{\mathcal{C}}$; moreover, $(-)^*$ is injective.*

Proof. Writing r_C for the right identity, one has a monoid homomorphism $(-)^* : \mathcal{C}(I, I) \longrightarrow [\mathcal{C}, \mathcal{C}](\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$, $f \mapsto (f^*_C : C \xrightarrow{r_C^{-1}} C \otimes I \xrightarrow{\mathrm{id}_C \otimes f} C \otimes I \xrightarrow{r_C} C)_{C \in \mathcal{C}}$ with right inverse $(-)_I : [\mathcal{C}, \mathcal{C}](\mathrm{id}, \mathrm{id}) \longrightarrow \mathcal{C}(I, I)$, $t \mapsto t_I$. So, $(-)^*$ is a monomorphism. This easily restricts to idempotents. □

So, natural transformations on $\mathrm{id}_{\mathcal{C}}$ limit the possible monoidal structures on $\mathcal{C}$. In particular,

Remark 4.9 For the category $\mathbf{X}$, there are exactly two idempotent natural transformations $\mathrm{id}, \alpha : \mathrm{id}_{\mathbf{X}} \longrightarrow \mathrm{id}_{\mathbf{X}}$ (Remark 3.13). So, for any monoidal structure on $\mathbf{X}$ whose unit has an idempotent on it, the unit is either the terminal object 1 or the initial object 0, since otherwise there would be more than two endomorphisms on the unit.

5 Presheaves

In this section, we take a small diagonal category $\mathcal{C}$, construct the presheaf category on it, and apply the construction of Proposition 3.9 to the idempotent $(t \otimes A) \cdot \delta_A$ to obtain a bireflective subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. The presheaf category is the free monoidal cocompletion of $\mathcal{C}$. We use this fact, together with the diagonal structure on $\mathcal{C}$, to deduce the relationship between the induced monoidal structure on $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and finite products in the bireflective subcategory. It follows that the latter is a full subcategory of the category of commutative monoids on $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

We only refer to $\mathbf{Set}$ as our base category in this section, whereas for our leading example, the base is the category of domains $\mathbf{D}$. All our results here extend to $\mathbf{D}$; in fact, they extend to any cartesian closed, complete and cocomplete category $\mathcal{V}$, if we start with a small diagonal $\mathcal{V}$-category $\mathcal{C}$. Every small diagonal category can be seen trivially as a small diagonal $\mathbf{D}$-category, so we can deduce results for our leading example immediately. We express our results only in terms of $\mathbf{Set}$ and ordinary categories merely for ease of exposition.

Theorem 5.1 (Im and Kelly [6]) *Let $\mathcal{C}$ be a small (symmetric) monoidal category. Then, the free (symmetric) monoidal cocompletion of $\mathcal{C}$ is $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ with (symmetric) monoidal structure given by left Kan extension*

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[r, "\mathcal{Y} \times \mathcal{Y}"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\cong"] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \times [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "\mathrm{Lan}_{\mathcal{Y}\times\mathcal{Y}}(\mathcal{Y}\otimes) = \widehat{\otimes}"] \\
\mathcal{C} \arrow[r, "\mathcal{Y}"'] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]}
\end{tikzcd}$$

□

Spelling this out, $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is cocomplete and $f \widehat{\otimes} -$ and $- \widehat{\otimes} f$ preserve colimits. An explicit formula for $\widehat{\otimes}$ is $(f \widehat{\otimes} g)C = \int^{X,Y \in \mathcal{C}} fX \times gY \times \mathcal{C}(C, X \otimes Y)$, *i.e.*, the coequalizer in $\mathbf{Set}$

$$\coprod_{\substack{X,Y,X',Y' \in \mathcal{C} \\ X' \xrightarrow{u} X \\ Y' \xrightarrow{v} Y}} fX \times gY \times \mathcal{C}(C, X' \otimes Y') \rightrightarrows \coprod_{X,Y \in \mathcal{C}} fX \times gY \times \mathcal{C}(C, X \otimes Y) \longrightarrow (f \widehat{\otimes} g)C$$

of the evident two maps $(X' \xrightarrow{u} X, Y' \xrightarrow{v} Y, x, y, w) \mapsto (f(u)x, g(v)y, w)$ and $(u, v, x, y, w) \mapsto (x, y, (u \otimes v)w)$.

Remark 5.2 The monoidal structure on the semantic category $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ given in Definition 2.5 agrees with the monoidal structure determined by Proposition 2.2 and the extension of Theorem 5.1 to $\mathbf{D}$ rather than $\mathbf{Set}$.

Now assume $\mathcal{C}$ is a diagonal category. We write α_C for the idempotent $(t \otimes C) \cdot \delta_C$. Let $\mathcal{C}/{\sim}$ be the coidentifier determined by α (Example 3.8). Consider the following diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "y"] \arrow[d, "h"'] & {[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, shift right, "S"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{C}/{\sim} \arrow[r, "y"'] & {[\mathcal{C}/{\sim}^{\mathrm{op}}, \mathbf{Set}]} \arrow[u, hook, shift right, "{[h, \mathbf{Set}]}"']
\end{tikzcd}$$

We have seen by Proposition 3.9 that $[h, \mathbf{Set}]$ is fully faithful with left and right adjoint equal, given by sending $f : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ to the splitting of $f\alpha$. Moreover, h sends the monoidal structure of $\mathcal{C}$ to finite product structure on $\mathcal{C}/{\sim}$ (Proposition 4.6). The category $[\mathcal{C}/{\sim}^{\mathrm{op}}, \mathbf{Set}]$ is cartesian closed and cocomplete. So, by the universal property of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, we have

Proposition 5.3 *Given the diagram above, S sends $\widehat{\otimes}$ on $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ to finite products on $[\mathcal{C}/{\sim}^{\mathrm{op}}, \mathbf{Set}]$.* □

Proposition 5.4 *For any $f, g \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, $(f \widehat{\otimes} g)\alpha = f\alpha \widehat{\otimes} g\alpha$.*

Proof. The families $((f \widehat{\otimes} g)\alpha)_{f,g \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]}$ and $(f\alpha \widehat{\otimes} g\alpha)_{f,g \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]}$ both form natural transformations from $\widehat{\otimes} : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \times [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ to itself. These two natural transformations are equal if and only if their restrictions under $y \times y$ to $\mathcal{C} \times \mathcal{C}$ are equal; this is immediate from the definition of $\widehat{\otimes}$ as a left Kan extension. So, it suffices to prove that for each Z in $\mathcal{C}$, and for each $X, Y \in \mathcal{C}$, the maps $\mathcal{C}(Z, \alpha_{X \otimes Y}) : \mathcal{C}(Z, X \otimes Y) \longrightarrow \mathcal{C}(Z, X \otimes Y)$ and $\mathcal{C}(Z, \alpha_X \otimes \alpha_Y)$ are equal; but that holds by a routine calculation using the third commutativity in the definition of diagonal category. □

We write $J : [\mathcal{C}/{\sim}^{\mathrm{op}}, \mathbf{Set}] \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ for the full inclusion $[h, \mathcal{C}]$.

Proposition 5.5 *For any $f, g \in [\mathcal{C}/{\sim}^{\mathrm{op}}, \mathbf{Set}]$, $Jf \widehat{\otimes} Jg = Jf \times Jg$ in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.*

Proof. We have $(Jf)\alpha = \mathrm{id}$ and $(Jg)\alpha = \mathrm{id}$, and we show first that $Jf \widehat{\otimes} Jg$ lies in $[\mathcal{C}/{\sim}^{\mathrm{op}}, \mathbf{Set}]$, *i.e.*, $(Jf \widehat{\otimes} Jg)\alpha = \mathrm{id}$; but this follows immediately from Proposition 5.4. So, $Jf \widehat{\otimes} Jg \cong JS(Jf \widehat{\otimes} Jg)$. Since S sends $\widehat{\otimes}$ to $\times$, and the right adjoint J preserves $\times$, we have

$$Jf \widehat{\otimes} Jg \cong JS(Jf \widehat{\otimes} Jg) = J(SJf \times SJg) \cong JSJf \times JSJg \cong Jf \times Jg.$$

□

Putting this together, we have

Theorem 5.6 *The full inclusion J sends finite products in $[\mathcal{C}/{\sim}^{\mathrm{op}}, \mathbf{Set}]$ to the monoidal structure of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, and has left and right adjoint sending f to the splitting of $f\alpha$, sending the symmetric monoidal structure to finite products. So, both $S \dashv J$ and $J \dashv S$ are monoidal adjunctions.* □

Corollary 5.7 *$[\mathcal{C}/{\sim}^{\mathrm{op}}, \mathbf{Set}]$ is a full subcategory of the category of commutative comonoids in $([\mathcal{C}^{\mathrm{op}}, \mathbf{Set}], \widehat{\otimes})$.*

Proof. Since $\widehat{\otimes}$ restricts to finite products on $[\mathcal{C}/{\sim}^{\mathrm{op}}, \mathbf{Set}]$, each object f of $[\mathcal{C}/{\sim}^{\mathrm{op}}, \mathbf{Set}]$ possesses a unique commutative comonoid structure. Fully faithfulness is obvious. $\Box$

In the particular case of Example 4.2, calculation of the formula for $\widehat{\otimes}$ reveals that $\mathbf{P}$ is precisely the category of commutative comonoids.

6 Closure

Finally, in this section, we address closed structure. None of our results here strictly requires the fact that we have a bireflective subcategory; in fact, they do not require we have presheaves either. However, the leading example is, as through the course of the paper, the inclusion of $\mathbf{P}$ into $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$. Recall from the previous section, that given a small symmetric monoidal category $\mathcal{C}$, the category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is the free symmetric monoidal cocompletion of $\mathcal{C}$. In fact, more is true: $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is symmetric monoidal closed. That result, together with all our analysis of the previous section, extends to small symmetric monoidal $\mathcal{V}$-categories, provided $\mathcal{V}$ is *locally presentable as a cartesian closed category* (see [7]). The category of domains is such a category, so for general reasons, $[\mathbf{X}^{\mathrm{op}}, \mathbf{D}]$ is symmetric monoidal closed.

To prove the results of this section, we consider a more general situation.

Proposition 6.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be symmetric monoidal closed, with $J{:}\mathcal{B} \longrightarrow \mathcal{A}$ a full inclusion with left adjoint F preserving symmetric monoidal structure up to coherent isomorphism. Then $\mathcal{B}$ is an exponential ideal of $\mathcal{A}$.*

Proof. It suffices to show that for any X in $\mathcal{B}$ and A in $\mathcal{A}$, $[A, JX]_{\mathcal{A}}$ lies in the image of J. To see that, apply Yoneda to the following sequence of natural isomorphims, for any C in $\mathcal{A}$:

$$\begin{aligned}\mathcal{A}(C, [A, JX]_{\mathcal{A}}) &\cong \mathcal{A}(C \otimes A, JX) \cong \mathcal{B}(F(C \otimes A), X)\\ &\cong \mathcal{B}(FC \otimes FA, X) \cong \mathcal{B}(FC, [FA, X]_{\mathcal{B}})\\ &\cong \mathcal{A}(C, J[FA, X]_{\mathcal{B}}).\end{aligned}$$

$\Box$

By a similar calculation, one can show that given any full coreflective subcategory $\mathcal{B}$ of a symmetric monoidal closed category $\mathcal{A}$ such that $\mathcal{B}$ is closed under the monoidal structure of $\mathcal{A}$, then $\mathcal{B}$ is symmetric monoidal closed. This allows us to deduce

Theorem 6.2 *Let $\mathcal{B}$ be a full reflective and coreflective subcategory of symmetric monoidal closed $\mathcal{A}$, and assume $\mathcal{B}$ is closed under the monoidal structure of $\mathcal{A}$ and the left adjoint preserves the symmetric monoidal structure. Then, $\mathcal{B}$ is symmetric monoidal closed and is in fact an exponential ideal of $\mathcal{A}$.* $\Box$

Putting this together with earlier results, we may conclude

Corollary 6.3 *Any small diagonal category $\mathcal{C}$ induces a bireflective subcategory $\mathcal{B}$ of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, such that $\mathcal{B}$ is a presheaf category, hence cartesian closed, and an exponential ideal in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, with both the inclusion and adjoint preserving the symmetric monoidal structure.* $\square$

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